MATH 2230 HW 6

(1) It follows from z^n having antiderivative in $\mathbb{C}$. $\forall n \in \mathbb{N}$

(2a) $\int_0^{1+i} z^2 dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{2}{3}(-1+i)$

(2b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = \left[2\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} = 2\left(\frac{e^{\frac{i\pi-2}{2}} - e^{\frac{-i\pi+2}{2}}}{2i} - 0\right)$

$= e + e^{-1}$

(2c) $\int_1^3 (z-2)^3 dz = \left[\frac{1}{4}(z-2)^4\right]_1^3 = 0.$

(3) It follows from $(z-z_0)^{n-1}$ having antiderivative in $\mathbb{C}$ $\forall n = \pm 1, \pm 2 \ldots$

(4) Since z^i is analytic in the upper half plane with the chosen branch. Its antiderivative is $\frac{1}{i+1} z^{i+1}$ in that domain,

$\int_{-1}^1 z^i dz = \left[\frac{1}{i+1} z^{i+1}\right]_{-1}^1 = \frac{1+e^{-\pi}}{2}(1-i)$.

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(5) a f is not analytic at where $3z^2+1=0$

$\Rightarrow z = \pm \frac{1}{\sqrt{3}} i$, which is inside C_1.

(b) f is not analytic at where $\sin\left(\frac{z}{2}\right)=0$

$\Rightarrow z = 2n\pi,\ n \in \mathbb{Z}$.

0 is inside C_1 and $2n\pi$ is outside C_2 $\forall$

$n \in \mathbb{Z} \setminus \{0\}$.

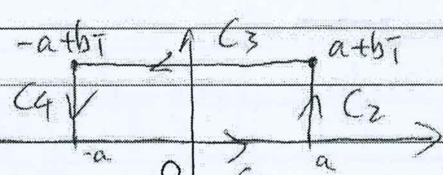
(c) f is not analytic at where $1-e^z=0$,

$\Rightarrow z = 0\ 2n\pi i\quad n \in \mathbb{Z}$, similar as (b).

(6) Consider a large enough circle C_0 that encloses and does intersect the rectangle. Since the only possible singularity $2+i$ is inside the rectangle, $(z-2-i)^{n-1}$ is analytic inside the region bounded by C and C_0. So

$$\int_C (z-2-i)^{n-1}\,dz = \int_{C_0} (z-2-i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \dots \\ 2\pi i & \text{when } n = 0 \end{cases}$$

(7a) On C_1, $e^{-z^2} = e^{-x^2}$



on C_3, $e^{-z^2} = e^{-(x+bi)^2}$

$= e^{-x^2+b^2-2xbi}$

$= e^{b^2}e^{-x^2}(\cos 2xb - \sin 2xb)$

$$\int_{C_1} e^{-z^2}dz + \int_{C_3} e^{-z^2}dz = \int_{-a}^{a} e^{-x^2}dx - \int_{-a}^{a} e^{b^2}e^{-x^2}(\cos 2xb - \sin 2xb)\,dx$$

$$= 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2xb,$$

Since e^{-x^2}, $e^{-x^2}\cos 2xb$ are even functions, and $e^{-x^2}\sin 2xb$ is an odd function.

7(a) On C_2, $e^{-z^2} = e^{-(a+yi)^2} = e^{-a^2+y^2-2ayi}$

On C_4, $e^{-z^2} = e^{-(-a+yi)^2} = e^{-a^2+y^2+2ayi}$

$$\int_{C_2} e^{-z^2}dz + \int_{C_4} e^{-z^2}dz = \int_0^b e^{-a^2+y^2-2ayi}(i)\,dy + \int_0^b e^{-a^2+y^2+2ayi}(-i)\,dy$$
$$= ie^{-a^2}\int_0^b e^{y^2}e^{-2ayi}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{2ayi}\,dy$$

7(b) Note that $\left|\int_{C_2}+\int_{C_4} e^{-z^2}dz\right| \le e^{-a^2}\left(\int_0^b |e^{y^2}e^{-2ayi}|\,dy + \int_0^b |e^{y^2}e^{2ayi}|\,dy\right)$

$$\le 2e^{-a^2}\int_0^b e^{y^2}\,dy \to 0 \quad \text{as } a \to +\infty$$

Let $C = C_1 + C_2 + C_3 + C_4$.

Since e^{-z^2} is an entire function,

$$\int_C e^{-z^2}\,dz = 0$$

So $\lim_{a\to\infty}\int_C e^{-z^2}dz = 0 \Rightarrow \underbrace{\lim_{a\to\infty}\left(\int_{C_2}+\int_{C_4}\right)e^{-z^2}dz}_{\to 0} + \lim_{a\to\infty}\left(\int_{C_1}+\int_{C_3}\right)e^{-z^2}dz = 0$

$$\Rightarrow \lim_{a\to\infty} 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx = 0$$
$$\Rightarrow \int_0^\infty e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^\infty e^{-x^2}\,dx$$
$$= \frac{\sqrt{\pi}}{2}e^{-b^2}$$